

**A WAVELET-BASED APPROXIMATION OF FRACTIONAL BROWNIAN MOTION WITH A PARALLEL ALGORITHM**DAWEI HONG,<sup>\*</sup> *Rutgers University*SHUSHUANG MAN,<sup>\*\*</sup> *Southwest Minnesota State University*JEAN-CAMILLE BIRGET,<sup>\*</sup> *Rutgers University*DESMOND S. LUN,<sup>\*</sup> *Rutgers University***Abstract**

We construct a wavelet-based expansion to approximate fractional Brownian motion of Hurst index  $H \in (0, 1)$ . For practical implementations, the expansion converges almost surely and uniformly in discrete time  $t \in [0, 1]$ . We prove that the convergence rate is optimal. We also show that the approximation can be implemented by a fast parallel algorithm.

*Keywords:* Fractional Brownian motion; wavelet expansion of stochastic integral; uniform approximation; convergence rate

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## 1. Introduction

A fractional Brownian motion (fBm)  $(B_t^{(H)})_{t \in [0, T]}$  of Hurst index  $H \in (0, 1)$  is a centered Gaussian process with covariance  $E[B_{t_1}^{(H)} B_{t_2}^{(H)}] = (1/2)(t_1^{2H} + t_2^{2H} - |t_1 - t_2|^{2H})$  for all  $t_1, t_2 \in [0, T]$ . A standard Brownian motion (Bm)  $(B_t)_{t \in [0, T]}$  is the special case of  $H = 1/2$ . There are enormous applications of fBm in engineering and sciences; see [3] and references therein. The study on approximation of fBm has been active since the 1970s. A major focus is to find approximations of fBm that converge in law; for example, see [6, 19, 5, 2, 15] and references therein. However, practical implementations often require almost sure and uniform approximations of fBm. Such an approximation works as follows. Let  $(B_t^{(H)})_{t \in [0, 1]}$  be a fBm of some  $H \in (0, 1)$ . Then, with respect to the probability space where  $(B_t^{(H)})_{t \in [0, 1]}$  is defined, the following event occurs with probability 1. For a sample path of  $(B_t^{(H)})_{t \in [0, 1]}$  there is a sequence of functions of  $t \in [0, 1]$  produced by the approximation which uniformly converges to the sample path; conversely, a sequence of functions of  $t \in [0, 1]$  produced by the approximation uniformly converges to a sample path of  $(B_t^{(H)})_{t \in [0, 1]}$ . Meyer, Sellan and Taqqu [17] obtained a wavelet series expansion of fBm which provides an almost sure and uniform approximation of fBm of  $H \in (0, 1)$ . The result also brings deep insights into spectral properties of fBm. The authors showed the existence of optimal wavelet expansion of fBm. However no convergence rates were given. Kühn and Linde [12] proved that the optimal convergence rate that any series expansion of fBm may reach is  $O(N^{-H} \sqrt{\log N})$  if the expansion converges uniformly over time interval  $[0, 1]$ . Dzharapadze and van Zanten [7] constructed an explicit series expansion of fBm of  $H \in (0, 1)$ . With probability 1 this series expansion converges absolutely and uniformly over time interval  $[0, 1]$ ; and soon after the authors proved that the convergence rate is optimal [8].

The above results will have a long lasting impact on the study of fBm. Nonetheless, there are two questions. The series expansion of fBm constructed in [7] is based on  $(\sin(x_n t)/x_n)_{n \geq 1}$  and  $(\cos(y_n t)/y_n)_{n \geq 1}$  where  $x_n$  and  $y_n$  are the positive zeros of the Bessel functions  $J_{-H}$  and  $J_{1-H}$  respectively. The wavelet series expansion of fBm in [17] uses the Haar wavelet; and in the remark on Theorem 2, the authors pointed out a technical difficulty to use the Mandelbrot - van Ness stochastic integral representation

of fBm [16] for wavelet series expansion of fBm. In practical implementations, the Haar wavelet is easier to compute than positive zeros of Bessel functions. Moreover, the simple form of the Mandelbrot - van Ness representation is likely to yield a fast algorithm. Thus, a question is if, using the Mandelbrot - van Ness representation, we can find a Haar wavelet series expansion of fBm of  $H \in (0, 1)$ , and prove that the expansion provides an almost sure and uniform approximation with the above optimal convergence rate.

In the present article, we construct a wavelet series expansion of fBm of  $H \in (0, 1)$  that meets the requirements mentioned above. Our method is to apply Lévy's equivalence theorem (see Theorem 2.6.2 in [13]) to a Haar wavelet expansion of fBm obtained based on the Mandelbrot - van Ness representation, and then to carefully evaluate the wavelet coefficients. We focus on discrete time  $t \in \mathbb{Q} \cap [0, 1]$  where  $\mathbb{Q}$  stands for the set of rational numbers. On one hand, the set of rational numbers covers all needs by practical implementations. On the other hand, by the Hölder continuity of fBm [1] with probability 1

$$\lim_{s \rightarrow 0^+} \sup \frac{|B_s^{(H)}|}{s^H \sqrt{\log \log s^{-1}}} = C_H \text{ for a constant } C_H > 0 \text{ depending only on } H$$

and hence, our result can be extended from the case of  $t \in \mathbb{Q} \cap [0, 1]$  to the case of  $t \in [0, 1]$ . Moreover, we show that our constructed wavelet series expansion of fBm yields a fast parallel algorithm for almost sure and uniform approximation of fBm of  $H \in (0, 1)$ .

Another question is if the constant behind the big  $O$  in the optimal convergence rate is related to Hurst index  $H$ . When a convergence rate for a series expansion of fBm is written as  $O(N^{-H} \sqrt{\log N})$ , the fact that fBm cannot be defined for  $H = 1$  cannot be seen. For our wavelet expansion of fBm, we show that the constant behind the big  $O$  is of the form  $C \sqrt{q} (H(1-H))^{-1/2}$  where  $C$  is absolute and  $q > 1$  is a parameter in the large deviation bound for the almost sure convergence; see Theorem 5.1 – the main result in this article.

Note that almost sure and uniform approximations of fBm were investigated by other approaches. One approach is by moving averages of simple random walks with convergence rate  $O(N^{-\log 4 \min\{H-1/4, 1/4\}} \log N)$  only for  $H \in (1/4, 1)$  [18]. Another approach is by transport processes with convergence rate  $O(N^{-(1/2-\beta)} (\log N)^{5/2})$ ,  $|H -$

$1/2| < \beta < 1/2$ , for  $H \in (0, 1)$  in [10]. The fastest convergence rate for almost sure and uniform approximations of fBm of  $H \in (0, 1)$  appears to be the optimal rate by series expansions of fBm.

The rest of the present article is organized as follows. Section 2 is for preliminaries. In sections 3, 4 and 5 we construct and prove a uniform approximation (in discrete time) of fBm of Hurst index  $H \in (0, 1)$ . In section 6 we present a parallel algorithm for the constructed approximation.

## 2. Preliminaries

The Mandelbrot - van Ness stochastic integral representation of fBm [16] is

$$B_t^{(H)} = C_H \int_{-\infty}^t \left( (t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right) dB_s, \quad t \in [0, 1]$$

Here  $H$  is the Hurst index taking values in  $(0, 1)$  and  $C_H = (\Gamma(H + 1/2))^{-1}$ , the reciprocal of the Gamma function at  $H + 1/2$ ; see 1.2 in [3] for results from further studies on the above representation of fBm. In what follows, we denote the underlying probability space for the above representation of fBm by  $(\Omega, \mathcal{F}, \Pr)$  where  $\mathcal{F}$  is a standard Brownian filtration.

Our construction of a uniform approximation of fBm is based on a rewriting of the Mandelbrot - van Ness stochastic integral representation

$$B_t^{(H)} = I_1(t, H) + I_2(t, H) + I_3(t, H), \quad t \in [0, 1] \quad (1)$$

where

$$\begin{aligned} I_1(t, H) &= C_H \int_0^t (t-s)^{H-1/2} dB_s \\ I_2(t, H) &= C_H \int_{-1}^0 \left( (t-s)^{H-1/2} - (-s)^{H-1/2} \right) dB_s \\ I_3(t, H) &= C_H \int_{-\infty}^{-1} \left( (t-s)^{H-1/2} - (-s)^{H-1/2} \right) dB_s \end{aligned}$$

Let  $(\phi_n)_{n \geq 0}$  be a complete orthonormal basis for  $L^2[a, b]$ . For  $f \in L^2[a, b]$ ,  $f = \sum_{n=0}^{\infty} \langle f, \phi_n \rangle \phi_n$  in  $L^2[a, b]$ . Take Wiener integration on both sides of the above equality, and informally interchange the order of integration and summation. We have

$$\int_a^b f(s) dB_s = \sum_{n=0}^{\infty} \langle f, \phi_n \rangle \int_a^b \phi_n(s) dB_s \quad (2)$$

We can check that the right side of (2) converges to the left side in  $E[|\cdot|^2]$ . However, by Lévy's equivalence theorem we further have the following (see Theorem 2.6.2 in [13]):

**Theorem 2.1.** *The right side of (2) converges almost surely, i.e., equality (2) holds with probability 1.  $\square$*

The Haar wavelet on  $[0, 1]$  is defined as follows. Let

$$\mathcal{H}(s) = \begin{cases} 1 & \text{if } 0 \leq s < 1/2 \\ -1 & \text{if } 1/2 \leq s \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For  $n = 2^j + k$  with  $j \geq 0$  and  $0 \leq k < 2^j$ , define  $\mathcal{H}_n(s) = 2^{j/2} \mathcal{H}(2^j s - k)$  and  $\mathcal{H}_0(s) = 1$ . The sequence  $(\mathcal{H}_n)_{n \geq 0}$  is the Haar wavelet on  $[0, 1]$ , which constitutes a complete orthonormal basis for  $L^2[0, 1]$ . In a similar way, we can define the Haar wavelet on any given interval  $[a, b] \subset \mathbb{R}$  to constitute a complete orthonormal basis for  $L^2[a, b]$  (see [4]).

### 3. Approximation of $I_1(t, H)$

In this and the next section, we construct and prove approximations of  $I_1(t, H)$  and  $I_2(t, H)$ , respectively. The two approximations are uniform over time  $t \in [0, 1] \cap \mathbb{Q}$ . In the meantime, we investigate how the convergence rates of these approximations are degrading when Hurst index  $H \rightarrow 1_-$  or  $0_+$ .

Consider a family of functions in  $L^2[0, 1]$ ,  $f_t^{(1)}$  with a parameter  $t \in (0, 1] \cap \mathbb{Q}$ . For every  $t \in (0, 1] \cap \mathbb{Q}$

$$f_t^{(1)}(s) = \begin{cases} (t - s)^{H-1/2} & \text{if } s \in [0, t) \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 2.1 we have

$$\Pr \left\{ \left( \int_0^1 f_t^{(1)}(s) dB_s \right) (\omega) = \left( \sum_{n=0}^{\infty} \langle f_t^{(1)}, \mathcal{H}_n \rangle \int_0^1 \mathcal{H}_n(s) dB_s \right) (\omega) \right\} = 1 \quad (3)$$

for each  $t \in (0, 1] \cap \mathbb{Q}$  and as a consequence

$$\Pr \bigcap_{t \in (0, 1] \cap \mathbb{Q}} \left\{ \left( \int_0^1 f_t^{(1)}(s) dB_s \right) (\omega) = \left( \sum_{n=0}^{\infty} \langle f_t^{(1)}, \mathcal{H}_n \rangle \int_0^1 \mathcal{H}_n(s) dB_s \right) (\omega) \right\} = 1.$$

We define for all  $N \geq 1$ ,

$$W_1(t, H, N) = \begin{cases} C_H \sum_{n=0}^N \langle f_t^{(1)}, \mathcal{H}_n \rangle \mathcal{L}_n^{(1)} & \text{for } t \in (0, 1] \cap \mathbb{Q} \\ 0 & \text{for } t = 0 \end{cases} \quad (4)$$

Here  $\mathcal{L}_n^{(1)} = \int_0^1 \mathcal{H}_n(s) dB_s$ ,  $n = 0, 1, \dots, N$ , are i.i.d. Gaussian random variables with mean 0 and variance 1.

In what follows we use two conventions:  $n \in \mathbb{Z}^+$  is said to be at level  $j$  if  $n = 2^j + k$  with  $j \geq 0$  and  $0 \leq k < 2^j$ ; and the interval  $[\frac{k}{2^j}, \frac{k+1}{2^j})$  is meant to be  $[\frac{k}{2^j}, \frac{k+1}{2^j}]$  when  $\frac{k+1}{2^j} = 1$ .

**Lemma 3.1.** *There is an absolute constant  $D_1 > 0$  such that for every  $t \in (0, 1] \cap \mathbb{Q}$  and for all  $N > 1$*

$$\sum_{n=N+1}^{\infty} \left\langle f_t^{(1)}, \mathcal{H}_n \right\rangle^2 \leq \frac{D_1}{H(1-H)N^{2H}}$$

**Proof.** For  $t \in (0, 1] \cap \mathbb{Q}$ , at each level  $j = 0, 1, \dots$ , we partition the set

$$\{n = 2^j + k : k = 0, 1, \dots, 2^j - 1\}$$

into three subsets:  $\mathcal{G}_1(j, t)$  consisting of all  $n$  (if any) such that  $t \geq \frac{k+1}{2^j}$ ;  $\mathcal{G}_2(j, t)$  consisting of one  $n$  such that  $t \in [\frac{k}{2^j}, \frac{k+1}{2^j})$ ; and  $\mathcal{G}_3(j, t)$  consisting of all  $n$  (if any) such that  $t < \frac{k}{2^j}$ .

Consider a fixed  $j$ . By the definition of  $f_t^{(1)}$  we have for  $n \in \mathcal{G}_3(j, t)$

$$\left\langle f_t^{(1)}, \mathcal{H}_n \right\rangle = 0 \quad (5)$$

Let us denote by  $2^j + \widehat{k_{t,j}}$  the unique  $n \in \mathcal{G}_2(j, t)$  such that  $t \in [\frac{\widehat{k_{t,j}}}{2^j}, \frac{\widehat{k_{t,j}}+1}{2^j})$ . Then we have  $n = 2^j + \widehat{k_{t,j}}$  and

$$\left\langle f_t^{(1)}, \mathcal{H}_n \right\rangle = 2^{j/2} \left[ \int_{\frac{2\widehat{k_{t,j}}}{2^{j+1}}}^{\frac{2\widehat{k_{t,j}}+1}{2^{j+1}}} f_t^{(1)}(s) ds - \int_{\frac{2\widehat{k_{t,j}}+1}{2^{j+1}}}^{\frac{2\widehat{k_{t,j}}+2}{2^{j+1}}} f_t^{(1)}(s) ds \right]$$

By the above equality we see that

$$\left| \left\langle f_t^{(1)}, \mathcal{H}_n \right\rangle \right| \leq 2^{j/2} \times \max \left\{ \int_{\frac{2\widehat{k}_{t,j}}{2^{j+1}}}^{\frac{2\widehat{k}_{t,j}+1}{2^{j+1}}} \left( \frac{2\widehat{k}_{t,j}+1}{2^{j+1}} - s \right)^{H-1/2} ds, \int_{\frac{2\widehat{k}_{t,j}+1}{2^{j+1}}}^{\frac{2\widehat{k}_{t,j}+2}{2^{j+1}}} \left( \frac{2\widehat{k}_{t,j}+2}{2^{j+1}} - s \right)^{H-1/2} ds \right\}$$

By this inequality, with calculation we have for  $n \in \mathcal{G}_2(j, t)$

$$\left\langle f_t^{(1)}, \mathcal{H}_n \right\rangle^2 \leq 2^{-2jH} \left( 2^{-(2H+1)} (H+1/2)^{-2} \right) \quad (6)$$

For each  $n \in \mathcal{G}_1(j, t)$ , we have  $n = 2^j + k$  with  $k < \widehat{k}_{t,j}$  and

$$\begin{aligned} \left\langle f_t^{(1)}, \mathcal{H}_n \right\rangle &= 2^{j/2} \left[ \int_{\frac{2k}{2^{j+1}}}^{\frac{2k+1}{2^{j+1}}} (t-s)^{H-1/2} ds - \int_{\frac{2k+1}{2^{j+1}}}^{\frac{2k+2}{2^{j+1}}} (t-s)^{H-1/2} ds \right] \\ &= \frac{2^{j/2}}{H+1/2} \left[ \left( \left( t - \frac{2k}{2^{j+1}} \right)^{H+1/2} - \left( t - \frac{2k+1}{2^{j+1}} \right)^{H+1/2} \right) - \right. \\ &\quad \left. \left( \left( t - \frac{2k+1}{2^{j+1}} \right)^{H+1/2} - \left( t - \frac{2k+2}{2^{j+1}} \right)^{H+1/2} \right) \right] \end{aligned} \quad (7)$$

To facilitate our argument, we introduce a function  $w$  of  $h$

$$w(h) = g(x_0 + h) + g(x_0 - h) - 2g(x_0)$$

where

$$g(\cdot) = (\cdot)^{H+1/2} \quad \text{and} \quad x_0 = t - \frac{2k+1}{2^{j+1}}$$

Then, letting  $h = \frac{1}{2^{j+1}}$ , we rewrite (7) as

$$\left\langle f_t^{(1)}, \mathcal{H}_n \right\rangle = \frac{2^{j/2}}{H+1/2} w(h) \quad (8)$$

Consider the following Taylor's expansion

$$\begin{aligned} w(h) &= w(0) + \frac{w'(0)}{1!} h + \frac{w''(\theta h)}{2!} h^2 \quad (\text{for some } 0 < \theta < 1) \\ &= \frac{w''(\theta h)}{2!} h^2 \quad (\text{since } w(0) = w'(0) = 0). \end{aligned}$$

Hence

$$\begin{aligned} w(h) &= h^2 \frac{w''(\theta h)}{2!} = 2^{-2(j+1)} \frac{(H+1/2)(H-1/2)}{2} \times \\ &\quad \left[ \left( t - \frac{2k+1+\theta}{2^{j+1}} \right)^{H-3/2} + \left( t - \frac{2k+1-\theta}{2^{j+1}} \right)^{H-3/2} \right] \end{aligned}$$

Now consider those  $n \in \mathcal{G}_1(j, t)$  with  $n = 2^j + k$  and  $k + 2 \leq \widehat{k_{t,j}}$ . By the above equality and (8) we have

$$\left| \langle f_t^{(1)}, \mathcal{H}_n \rangle \right| \leq 2^{j/2} 2^{-2(j+1)} |H - 1/2| \left( t - \frac{2k+1+\theta}{2^{j+1}} \right)^{H-3/2}$$

since  $0 < \theta < 1$  and  $0 < H < 1$ . Furthermore, since  $t \in [\frac{\widehat{k_{t,j}}}{2^j}, \frac{\widehat{k_{t,j}}+1}{2^j})$ , by the above equality we have

$$\begin{aligned} \left| \langle f_t^{(1)}, \mathcal{H}_n \rangle \right| &\leq 2^{j/2} 2^{-2(j+1)} |H - 1/2| \left( \frac{2\widehat{k_{t,j}}}{2^{j+1}} - \frac{2k+2}{2^{j+1}} \right)^{H-3/2} \\ &= \frac{|H - 1/2|}{4} 2^{-jH} \left( \widehat{k_{t,j}} - (k+1) \right)^{H-3/2} \end{aligned}$$

Thus, for  $n \in \mathcal{G}_1(j, t)$  with  $n = 2^j + k$  and  $k + 2 \leq \widehat{k_{t,j}}$ , we have

$$\left| \langle f_t^{(1)}, \mathcal{H}_n \rangle \right|^2 \leq 2^{-2jH} \left( \widehat{k_{t,j}} - (k+1) \right)^{2H-3} \frac{|H - 1/2|^2}{16} \quad (9)$$

There is one and only one  $\langle f_t^{(1)}, \mathcal{H}_n \rangle$  with  $n \in \mathcal{G}_1(j, t)$  which is not included in (9), namely the case of  $n = 2^j + \widehat{k_{t,j}} - 1$ . However, in this case we have

$$\langle f_t^{(1)}, \mathcal{H}_n \rangle = 2^{j/2} \left[ \int_{\frac{2\widehat{k_{t,j}}-2}{2^{j+1}}}^{\frac{2\widehat{k_{t,j}}-1}{2^{j+1}}} (t-s)^{H-1/2} ds - \int_{\frac{2\widehat{k_{t,j}}-1}{2^{j+1}}}^{\frac{2\widehat{k_{t,j}}}{2^{j+1}}} (t-s)^{H-1/2} ds \right]$$

and hence

$$\left| \langle f_t^{(1)}, \mathcal{H}_n \rangle \right|^2 \leq \frac{2^j}{(H+1/2)^2} 2^{-(2H+1)j} = 2^{-2jH} (H+1/2)^{-2} \quad (10)$$

Now, putting (5), (6), (9) and (10) together, we have that there is an absolute constant  $D_1 > 0$  such that at any level  $j$

$$\begin{aligned} \sum_{\{n \text{ at level } j\}} \left| \langle f_t^{(1)}, \mathcal{H}_n \rangle \right|^2 &\leq D_1 2^{-2jH} \sum_{\ell=1}^{\infty} \left( \frac{1}{\ell} \right)^{3-2H} \\ &= D_1 2^{-2jH} \left( 1 + \sum_{\ell=2}^{\infty} \left( \frac{1}{\ell} \right)^{3-2H} \right) \leq D_1 2^{-2jH} \left( 1 + \int_1^{\infty} \frac{dv}{v^{3-2H}} \right) \end{aligned}$$

Without loss of generality, we can rewrite the above inequality as

$$\sum_{\{n \text{ at level } j\}} \left| \langle f_t^{(1)}, \mathcal{H}_n \rangle \right|^2 \leq \frac{D_1}{1-H} 2^{-2jH}$$



Therefore we have

$$\begin{aligned} \sum_{n=N+1}^{\infty} \langle f_t^{(1)}, \mathcal{H}_n \rangle^2 &\leq \sum_{j=\lfloor \log_2 N \rfloor}^{\infty} \sum_{\{n \text{ at level } j\}} \left| \langle f_t^{(1)}, \mathcal{H}_n \rangle \right|^2 \leq \sum_{j=\lfloor \log_2 N \rfloor}^{\infty} \frac{D_1}{1-H} 2^{-2jH} \\ &= \frac{D_1}{1-H} 2^{-2\lfloor \log_2 N \rfloor H} \sum_{j=0}^{\infty} 2^{-2jH} = \frac{D_1}{1-H} 2^{-2\lfloor \log_2 N \rfloor H} \frac{1}{1-2^{-2H}} \end{aligned}$$

Since  $\lim_{H \rightarrow 0+} (1-2^{-2H})/H = 2 \log 2$ , there is an absolute constant  $G > 0$  such that  $1/(1-2^{-2H}) \leq G/H$  for all  $H \in (0, 1)$ . Hence we can rewrite the above inequality as claimed by the lemma.  $\square$

**Lemma 3.2.** *For any given  $H \in (0, 1)$  and  $q > 1$ , we have for all  $N > 1$*

$$\Pr \left\{ \sup_{t \in [0, 1] \cap \mathbb{Q}} |I_1(t, H) - W_1(t, H, N)| \geq \frac{C_H \sqrt{2D_1 q}}{\sqrt{H(1-H)}} \frac{\sqrt{\log N}}{N^H} \right\} \leq \frac{1}{\sqrt{q} N^q}$$

where  $D_1$  is the absolute constant used in Lemma 3.1.

**Proof.** By definition  $I_1(0, H) = 0 = W_1(0, H, N)$ . So, we focus on the case of  $t \in (0, 1] \cap \mathbb{Q}$ . By (4), (3) and its consequence we have

$$\begin{aligned} \Pr \bigcap_{t \in (0, 1] \cap \mathbb{Q}} \{ (I_1(t, H) - W_1(t, H, N))(\omega) = \\ C_H \sum_{n=N+1}^{\infty} \langle f_t^{(1)}, \mathcal{H}_n \rangle \int_0^1 \mathcal{H}_n(s) dB_s(\omega) \} = 1. \end{aligned} \quad (11)$$

Here  $\sum_{n=N+1}^{\infty} \langle f_t^{(1)}, \mathcal{H}_n \rangle \int_0^1 \mathcal{H}_n(s) dB_s$  is a Gaussian random variable with mean 0 and variance  $\sum_{n=N+1}^{\infty} \langle f_t^{(1)}, \mathcal{H}_n \rangle^2$ . We denote this variance by  $\sigma_1^2(t, H, N)$ .

For any given  $H \in (0, 1)$  and  $q > 1$ , we have

$$\begin{aligned} \Pr \left\{ \left| \sum_{n=N+1}^{\infty} \int_0^1 \langle f_t^{(1)}, \mathcal{H}_n \rangle \mathcal{H}_n(s) dB_s \right| > \frac{\sqrt{2D_1 q \log N}}{N^H \sqrt{H(1-H)}} \right\} \\ = \sqrt{\frac{2}{\pi}} \int_{\frac{\sqrt{2D_1 q \log N}}{N^H \sqrt{H(1-H)}}}^{\infty} \exp \left( -\frac{u^2}{2\sigma_1^2(t, H, N)} \right) du \\ \leq \frac{2\sigma_1(t, H, N)}{\sqrt{\pi}} \times \\ \int_{\frac{\sqrt{2D_1 q \log N}}{N^H \sqrt{H(1-H)}}}^{\infty} \frac{N^H \sqrt{H(1-H)}}{\sqrt{2D_1 q \log N}} \left( \frac{u}{\sqrt{2}\sigma_1(t, H, N)} \right) \exp \left( -\left( \frac{u}{\sqrt{2}\sigma_1(t, H, N)} \right)^2 \right) du \\ = \frac{\sigma_1(t, H, N)}{\sqrt{\pi}} \times \frac{N^H \sqrt{H(1-H)}}{\sqrt{2D_1 q \log N}} \times \exp \left( -\frac{D_1 q \log N}{\sigma_1^2(t, H, N) N^{2H} H(1-H)} \right) \\ \leq \frac{1}{\sqrt{2\pi q \log N}} \times \frac{1}{N^q} \text{ by Lemma 3.1.} \end{aligned}$$

By inequality (11) and since  $1/\sqrt{2\pi\log N} < 1$  for all  $N > 1$ , we have completed the proof of the Lemma.  $\square$

A remark for the above Lemma is as follows. By Lemma 3.2 and the Borel-Cantelli Lemma, we can see that when  $N \rightarrow \infty$ ,  $W_1(t, H, N)$  converges to  $I_1(t, H)$  almost surely and uniformly in  $t \in [0, 1] \cap \mathbb{Q}$ . In the meantime we also see that due to the factor  $1/\sqrt{H(1-H)}$  the convergence rate may degrade as the Hurst index  $H \rightarrow 1_-$  or  $0_+$ . The result in [12] shows that the optimal convergence rate is  $O(N^{-H}\sqrt{\log N})$ . Hence the above factor causes a degradation of the convergence rate when  $H \rightarrow 1_-$  or  $0_+$ .

#### 4. Approximation of $I_2(t, H)$

The construction of an approximation of  $I_2(t, H)$  follows a procedure similar to that in the previous section. Consider the Haar wavelet  $(\tilde{\mathcal{H}}_n)_{n \geq 0}$  on  $[-1, 0]$ . Define a family of functions in  $L^2[-1, 0]$ ,  $f_t^{(2)}$  with a parameter  $t \in [0, 1] \cap \mathbb{Q}$ :

$$f_t^{(2)}(s) = \begin{cases} (t-s)^{H-1/2} - (-s)^{H-1/2} & \text{if } s \in [-1, 0), \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 2.1 we have

$$\Pr \left\{ \left( \int_{-1}^0 f_t^{(2)}(s) dB_s \right) (\omega) = \left( \sum_{n=0}^{\infty} \langle f_t^{(2)}, \tilde{\mathcal{H}}_n \rangle \int_{-1}^0 \tilde{\mathcal{H}}_n(s) dB_s \right) (\omega) \right\} = 1 \quad (12)$$

for each  $t \in [0, 1] \cap \mathbb{Q}$ , and as a consequence

$$\Pr \bigcap_{t \in [0, 1] \cap \mathbb{Q}} \left\{ \left( \int_{-1}^0 f_t^{(2)}(s) dB_s \right) (\omega) = \left( \sum_{n=0}^{\infty} \langle f_t^{(2)}, \tilde{\mathcal{H}}_n \rangle \int_{-1}^0 \tilde{\mathcal{H}}_n(s) dB_s \right) (\omega) \right\} = 1.$$

We define for all  $N \geq 1$ ,

$$W_2(t, H, N) = \begin{cases} C_H \sum_{n=0}^N \langle f_t^{(2)}, \tilde{\mathcal{H}}_n \rangle \mathcal{L}_n^{(2)} & \text{for } t \in [0, 1] \cap \mathbb{Q} \\ 0 & \text{for } t = 0 \end{cases} \quad (13)$$

Here  $\mathcal{L}_n^{(2)} = \int_{-1}^0 \tilde{\mathcal{H}}_n(s) dB_s$ ,  $n = 0, 1, \dots, N$ , are i.i.d. Gaussian random variables with mean 0 and variance 1. Notice that the sequence  $(\mathcal{L}_n^{(2)})_{n \geq 0}$  is independent of the sequence  $(\mathcal{L}_n^{(1)})_{n \geq 0}$  used in the definition of  $W_1(t, H, N)$ .

**Lemma 4.1.** *There is an absolute constant  $D_2 > 0$  such that for every  $t \in [0, 1] \cap \mathbb{Q}$  and for all  $N > 1$*

$$\sum_{n=N+1}^{\infty} \left\langle f_t^{(2)}, \tilde{\mathcal{H}}_n \right\rangle^2 \leq \frac{D_2}{H(1-H)N^{2H}}$$

**Proof.** For each  $t \in [0, 1] \cap \mathbb{Q}$

$$\begin{aligned} & \sum_{n=N+1}^{\infty} \left\langle f_t^{(2)}, \tilde{\mathcal{H}}_n \right\rangle^2 \\ & \leq 2 \left( \sum_{n=N+1}^{\infty} \left\langle (t-s)^{H-1/2}, \tilde{\mathcal{H}}_n \right\rangle^2 + \sum_{n=N+1}^{\infty} \left\langle (-s)^{H-1/2}, \tilde{\mathcal{H}}_n \right\rangle^2 \right) \end{aligned} \quad (14)$$

Here  $\langle (t-s)^{H-1/2}, \tilde{\mathcal{H}}_n \rangle$  and  $\langle (-s)^{H-1/2}, \tilde{\mathcal{H}}_n \rangle$  are Riemann integrals. Thus, instead of  $\langle (t-s)^{H-1/2}, \tilde{\mathcal{H}}_n \rangle$  and  $\langle (-s)^{H-1/2}, \tilde{\mathcal{H}}_n \rangle$ , by changing variables we can estimate the right side of the above inequality, by using  $\langle (t+s)^{H-1/2}, \mathcal{H}_n \rangle$  and  $\langle s^{H-1/2}, \mathcal{H}_n \rangle$ . Below, we estimate  $\sum_{n=N+1}^{\infty} \langle (t+s)^{H-1/2}, \mathcal{H}_n \rangle^2$  and  $\sum_{n=N+1}^{\infty} \langle s^{H-1/2}, \mathcal{H}_n \rangle^2$  separately.

We estimate the first term. For  $t \in (0, 1] \cap \mathbb{Q}$ , at each level  $j$ , we have for each  $n = 2^j + k$ ,  $k = 0, \dots, 2^j - 1$ ,

$$\begin{aligned} & \left\langle (t+s)^{H-1/2}, \mathcal{H}_n \right\rangle \\ & = 2^{j/2} \left[ \int_{\frac{2k}{2^{j+1}}}^{\frac{2k+1}{2^{j+1}}} (t+s)^{H-1/2} ds - \int_{\frac{2k+1}{2^{j+1}}}^{\frac{2k+2}{2^{j+1}}} (t+s)^{H-1/2} ds \right] \\ & = \frac{2^{j/2}}{H+1/2} \left[ \left( \left( t + \frac{2k+1}{2^{j+1}} \right)^{H+1/2} - \left( t + \frac{2k}{2^{j+1}} \right)^{H+1/2} \right) - \right. \\ & \quad \left. \left( \left( t + \frac{2k+2}{2^{j+1}} \right)^{H+1/2} - \left( t + \frac{2k+1}{2^{j+1}} \right)^{H+1/2} \right) \right] \end{aligned} \quad (15)$$

To facilitate our argument we introduce a new function  $w$  of  $h$  which is a revised version of the function  $w$  in the proof of Lemma 3.1:

$$w(h) = 2g(x_0) - g(x_0 + h) - g(x_0 - h)$$

where

$$g(\cdot) = (\cdot)^{H+1/2} \quad \text{and} \quad x_0 = t + \frac{2k+1}{2^{j+1}}.$$

Then, letting  $h = \frac{1}{2^{j+1}}$ , we rewrite (15) as

$$\left\langle (t+s)^{H-1/2}, \mathcal{H}_n \right\rangle = \frac{2^{j/2}}{H+1/2} w(h)$$

Then, similarly to the proof of Lemma 3.1, we consider the following Taylor expansion:

$$\begin{aligned} w(h) &= w(0) + \frac{w'(0)}{1!}h + \frac{w''(\theta h)}{2!}h^2 \quad (\text{for some } 0 < \theta < 1) \\ &= \frac{w''(\theta h)}{2!}h^2 \quad (\text{since } w(0) = w'(0) = 0). \end{aligned}$$

Hence we have

$$\begin{aligned} w(h) &= h^2 \frac{w''(\theta h)}{2!} = -2^{-2(j+1)} \frac{(H+1/2)(H-1/2)}{2} \times \\ &\quad \left[ \left( t + \frac{2k+1+\theta}{2^{j+1}} \right)^{H-3/2} + \left( t + \frac{2k+1-\theta}{2^{j+1}} \right)^{H-3/2} \right] \end{aligned}$$

and consequently

$$\left| \left\langle (t+s)^{H-1/2}, \mathcal{H}_n \right\rangle \right| \leq 2^{j/2} 2^{-2(j+1)} |H-1/2| \left( t + \frac{2k+1-\theta}{2^{j+1}} \right)^{H-3/2} \quad (16)$$

since  $0 < \theta < 1$  and  $0 < H < 1$ . Now, as in the proof of Lemma 3.1, we denote by  $2^j + \widehat{k_{t,j}}$  the unique  $n$  such that  $t \in [\frac{\widehat{k_{t,j}}}{2^j}, \frac{\widehat{k_{t,j}}+1}{2^j})$ . Then, there are two and only two cases.

*Case 1:*  $\widehat{k_{t,j}} \geq 1$ . By (16) we have

$$\begin{aligned} &\left| \left\langle (t+s)^{H-1/2}, \mathcal{H}_n \right\rangle \right| \\ &\leq 2^{j/2} 2^{-2(j+1)} |H-1/2| \left( \frac{\widehat{2k_{t,j}}}{2^{j+1}} + \frac{2k+1-\theta}{2^{j+1}} \right)^{H-3/2} \\ &\leq 2^{-2jH} |H-1/2| 2^{-(H+1/2)} (k+1)^{H-3/2} \end{aligned}$$

Thus, there is an absolute constant  $D_{2,1} > 0$  such that

$$\sum_{\{n \text{ at level } j\}} \left| \left\langle (t+s)^{H-1/2}, \mathcal{H}_n \right\rangle \right|^2 \leq D_{2,1} 2^{-2jH} \sum_{\ell=1}^{\infty} \left( \frac{1}{\ell} \right)^{3-2H} \quad (17)$$

*Case 2:*  $\widehat{k_{t,j}} = 0$ . Using (15) we have

$$\begin{aligned} &\left\langle (t+s)^{H-1/2}, \mathcal{H}_{2^j} \right\rangle \\ &= 2^{j/2} \left[ \int_0^{\frac{1}{2^{j+1}}} (t+s)^{H-1/2} ds - \int_{\frac{1}{2^{j+1}}}^{\frac{2}{2^{j+1}}} (t+s)^{H-1/2} ds \right] \\ &= \frac{2^{j/2}}{H+1/2} \left[ \left( \left( t + \frac{1}{2^{j+1}} \right)^{H+1/2} - t^{H+1/2} \right) - \right. \\ &\quad \left. \left( \left( t + \frac{2}{2^{j+1}} \right)^{H+1/2} - \left( t + \frac{1}{2^{j+1}} \right)^{H+1/2} \right) \right] \end{aligned}$$

and hence

$$\left| \left\langle (t+s)^{H-1/2}, \mathcal{H}_{2^j} \right\rangle \right| \leq \frac{2^{j/2}}{H+1/2} \left( \frac{2}{2^j} \right)^{H+1/2} \quad (18)$$

For  $n = 2^j + k$  with  $k = 1, \dots, 2^j - 1$ , by (16) we have

$$\begin{aligned} \left| \left\langle (t+s)^{H-1/2}, \mathcal{H}_n \right\rangle \right| &\leq 2^{j/2} 2^{-2(j+1)} |H-1/2| \left( \frac{2k+1-\theta}{2^{j+1}} \right)^{H-3/2} \\ &< \frac{2^{j/2}}{H+1/2} 2^{-2(j+1)} \left( \frac{k}{2^j} \right)^{H-3/2} \quad \text{since } |H^2 - 1/4| < 1 \text{ for } H \in (0, 1). \end{aligned} \quad (19)$$

Putting (18) and (19) together, we have

$$\sum_{\{n \text{ at level } j\}} \left\langle (t+s)^{H-1/2}, \mathcal{H}_n \right\rangle^2 \leq D_{2,1} 2^{-2jH} \sum_{\ell=1}^{\infty} \left( \frac{1}{\ell} \right)^{3-2H} \quad (20)$$

Here, without loss of generality we can let  $D_{2,1}$  be the same absolute constant as in (17).

With an argument similar to the above for  $\langle (t+s)^{H-1/2}, \mathcal{H}_n \rangle$ , we have that there is an absolute constant  $D_{2,2} > 0$  such that

$$\sum_{\{n \text{ at level } j\}} \left\langle s^{H-1/2}, \mathcal{H}_n \right\rangle^2 \leq D_{2,2} 2^{-2jH} \sum_{\ell=1}^{\infty} \left( \frac{1}{\ell} \right)^{3-2H} \quad (21)$$

The lemma follows from putting (14), (20) and (21) together.  $\square$

**Lemma 4.2.** *For any given  $H \in (0, 1)$  and  $q > 1$ , we have all  $N > 1$*

$$\Pr \left\{ \sup_{t \in [0, 1] \cap \mathbb{Q}} |I_2(t, H) - W_2(t, H, N)| \geq \frac{C_H \sqrt{2D_2 q}}{\sqrt{H(1-H)}} \frac{\sqrt{\log N}}{N^H} \right\} \leq \frac{2}{\sqrt{q} N^q}$$

where  $D_2$  is the absolute constant used in Lemma 4.1.

**Proof.** By (13), (12) and its consequence we have

$$\begin{aligned} \Pr \bigcap_{t \in [0, 1] \cap \mathbb{Q}} \{ (I_2(t, H) - W_2(t, H, N))(\omega) = \\ C_H \sum_{n=N+1}^{\infty} \langle f_t^{(2)}, \tilde{\mathcal{H}}_n \rangle \int_{-1}^0 \tilde{\mathcal{H}}_n(s) dB_s(\omega) \} = 1 \end{aligned}$$

Here,  $\sum_{n=N+1}^{\infty} \langle f_t^{(2)}, \tilde{\mathcal{H}}_n \rangle \int_{-1}^0 \mathcal{H}_n(s) dB_s$  is a Gaussian random variable with mean 0 and variance  $\sum_{n=N+1}^{\infty} \langle f_t^{(2)}, \tilde{\mathcal{H}}_n \rangle^2$ . By Lemma 4.1, following the same procedure as in the proof of Lemma 3.2, we complete a proof of the lemma.  $\square$

The result of Lemma 4.2 has an expression similar to that of Lemma 3.2. Thus the remark made there holds here.

### 5. Approximation of fBm

For  $H \in (0, 1)$ , given  $0 \leq t_1 < \dots < t_i < \dots < t_\ell \leq 1$  with all  $t_i \in \mathbb{Q}$ , find a sequence  $(W(t, H, N))_{N \geq 1}$  of functions defined on  $t \in [0, 1]$  such that with probability 1 the above sequence converges to a sample path of  $(B_t^{(H)})_{t \in [0, 1]}$  uniformly over  $t = t_i$ ,  $i = 1, \dots, \ell$ . For a given  $N$ ,  $W(t, H, N)$  is called the  $N$ th step of a uniform in discrete time approximation of  $(B_t^{(H)})_{t \in [0, 1]}$ , or simply the  $N$ th approximation of  $(B_t^{(H)})_{t \in [0, 1]}$  when all  $t_i$  are regarded as predetermined. With this in mind, let us take a close look at  $(B_t^{(H)})_{t \in [0, 1]}$  represented by (1).

$I_1(t, H)$  is the process  $\left(C_H \int_0^t (t-s)^{H-1/2} dB_s\right)_{t \in [0, 1]}$ . At each time  $t \in [0, 1]$  the process is defined by an Itô integral  $C_H \int_0^t (t-s)^{H-1/2} dB_s$  which itself is a process on  $[0, t]$ . We considered

$$\left(C_H \int_0^t (t-s)^{H-1/2} dB_s\right)_{t \in [0, 1] \cap \mathbb{Q}} \quad (22)$$

and defined  $W_1(t, H, N)$  that converges to (22) almost surely and uniformly in  $t \in \mathbb{Q} \cap [0, 1]$ . Here  $W_1(t, H, N)$  is defined via  $\mathcal{L}_n^{(1)} = \int_0^1 \mathcal{H}_n(s) dB_s$ ,  $n = 0, 1, \dots$ , and hence, it is defined on  $(\Omega, \mathcal{F}, \text{Pr})$ .

$I_2(t, H)$  is the process  $\left(C_H \int_{-1}^0 ((t-s)^{H-1/2} - (-s)^{H-1/2}) dB_s\right)_{t \in [0, 1]}$ . This process is indexed by time in the usual way, i.e., at each time  $t \in [0, 1]$  the process is defined by the random variable  $C_H \int_{-1}^0 ((t-s)^{H-1/2} - (-s)^{H-1/2}) dB_s$  (not an Itô integral). Thus

$$\left(C_H \int_{-1}^0 ((t-s)^{H-1/2} - (-s)^{H-1/2}) dB_s\right)_{t \in [0, 1] \cap \mathbb{Q}} \quad (23)$$

is a process defined on  $(\Omega, \mathcal{F}, \text{Pr})$ . Because the integrand tends to infinity as  $s \rightarrow 0_-$ , in the case of  $H \in (0, 1/2)$  we defined  $W_2(t, H, N)$  that converges to (23) almost surely and uniformly in  $t \in \mathbb{Q} \cap [0, 1]$ . Here  $W_2(t, H, N)$  was defined via  $\mathcal{L}_n^{(2)} = \int_{-1}^0 \mathcal{H}_n(s) dB_s$ ,  $n = 0, 1, \dots$ , and hence, it is defined on  $(\Omega, \mathcal{F}, \text{Pr})$ .

$I_3(t, H)$  is the process  $\left(C_H \int_{-\infty}^{-1} ((t-s)^{H-1/2} - (-s)^{H-1/2}) dB_s\right)_{t \in [0, 1]}$ . This process is indexed by time in the usual way, i.e., at each time  $t \in [0, 1]$  the process is defined by the random variable  $C_H \int_{-\infty}^{-1} ((t-s)^{H-1/2} - (-s)^{H-1/2}) dB_s$  (not an Itô integral). Thus

$$\left(C_H \int_{-\infty}^{-1} ((t-s)^{H-1/2} - (-s)^{H-1/2}) dB_s\right)_{t \in [0, 1] \cap \mathbb{Q}} \quad (24)$$

is a process defined on  $(\Omega, \mathcal{F}, \Pr)$ . Since the integrand is well defined in  $s \in (-\infty, -1]$ , we have a direct simulation (not an approximation) of (24). Combining this simulation of (24) with  $W_1(t, H, N)$  and  $W_2(t, H, N)$ , we obtain an approximation of  $(B_t^{(H)})_{t \in [0,1] \cap \mathbb{Q}}$ . Below we carry out this idea.

For a fixed  $t \in [0, 1] \cap \mathbb{Q}$ ,  $C_H \int_{-\infty}^{-1} ((t-s)^{H-1/2} - (-s)^{H-1/2}) dB_s$  generates a Gaussian random variable  $X(t)$  with mean 0 and variance

$$C_H^2 \int_{-\infty}^{-1} \left( (t-s)^{H-1/2} - (-s)^{H-1/2} \right)^2 ds$$

as follows. For almost every  $\omega \in \Omega$

$$X(t)(\omega) = C_H \int_{-\infty}^{-1} \left( (t-s)^{H-1/2} - (-s)^{H-1/2} \right) dB_s(\omega) \quad (25)$$

By definition  $X(t)$  is independent of  $\mathcal{L}_n^{(1)}$  and  $\mathcal{L}_n^{(2)}$ ,  $n = 0, 1, \dots$ , in the definition of  $W_1(t, H, N)$  and  $W_2(t, H, N)$ . By (25) we have

$$\Pr \bigcap_{t \in [0,1] \cap \mathbb{Q}} \left\{ C_H \int_{-\infty}^{-1} \left( (t-s)^{H-1/2} - (-s)^{H-1/2} \right) dB_s(\omega) = \left( C_H \left( \int_{-\infty}^{-1} \left( (t-s)^{H-1/2} - (-s)^{H-1/2} \right)^2 ds \right)^{1/2} \right) \mathcal{L}(\omega) \right\} = 1$$

where  $\mathcal{L}$  is a standard Gaussian random variable independent of  $\mathcal{L}_n^{(1)}$  and  $\mathcal{L}_n^{(2)}$ ,  $n = 0, 1, \dots$ . Hence, given  $t_i \in [0, 1] \cap \mathbb{Q}$ ,  $i = 1, \dots, \ell$ , we can directly simulate  $I_3(t_i, H)$  as follows. Generate a sample  $\mathcal{L}(\omega)$  of a standard Gaussian random variable independent of  $\mathcal{L}_n^{(1)}$  and  $\mathcal{L}_n^{(2)}$ ,  $n = 0, 1, \dots$ ; then for  $i = 1, \dots, \ell$ , let

$$I_3(t_i, H)(\omega) = \left( C_H \left( \int_{-\infty}^{-1} \left( (t_i-s)^{H-1/2} - (-s)^{H-1/2} \right)^2 ds \right)^{1/2} \right) \mathcal{L}(\omega) \quad (26)$$

Now, in  $(\Omega, \mathcal{F}, \Pr)$  we define

$$W(t, H, N) = W_1(t, H, N) + W_2(t, H, N) + I_3(t, H) \quad \text{for } t \in [0, 1] \cap \mathbb{Q}$$

**Theorem 5.1.** *There is an absolute constant  $C > 0$  such that, for any given  $H \in (0, 1)$  and  $q > 1$ , we have for all  $N > 1$*

$$\Pr \left\{ \sup_{t \in [0,1] \cap \mathbb{Q}} \left| B_t^{(H)} - W(t, H, N) \right| \geq \frac{C\sqrt{q}}{\sqrt{H(1-H)}} \frac{\sqrt{\log N}}{N^H} \right\} \leq \frac{3}{\sqrt{q}N^q}$$

**Proof.** Using  $2N + 3$  independent samples from the standard Gaussian distribution, we can approximate a path in a fBm  $(B_t^{(H)})_{t \in [0,1]}$  for any given  $t_i \in [0, 1] \cap \mathbb{Q}$ ,  $i = 1, \dots, \ell$ , as follows. We use the first  $N + 1$  samples to obtain  $\ell$  instances of  $W_1(t, H, N)$  at  $t = t_i$  according to (4), and use the next  $N + 1$  samples to obtain  $\ell$  instances of  $W_2(t, H, N)$  at  $t = t_i$  according to (13). By definition, these  $2\ell$  instances approximate a common path in  $(B_t^{(H)})_{t \in [0,1]}$ . We use the last sample to obtain  $\ell$  instances of  $I_3(t, H)$  at  $t = t_i$  according to (26). By definition, the obtained  $\ell$  instances belong to the path in  $(B_t^{(H)})_{t \in [0,1]}$  which is approximated by using the first  $2N + 2$  samples as described above. Then the theorem follows from Lemma 3.2 and 4.2, (26), and the fact that  $\ell$  can be any number in  $\mathbb{N}$ .

Before leaving the proof, we note that the variance in (26) is bounded for  $t \in [0, 1] \cap \mathbb{Q}$ . Indeed,

$$\begin{aligned} \left| (t+s)^{H-1/2} - s^{H-1/2} \right| &= |H-1/2| \int_0^t (u+s)^{H-3/2} du \\ &\leq |H-1/2| s^{H-3/2} \int_0^t du = t|H-1/2| s^{H-3/2} \end{aligned}$$

and hence,

$$\begin{aligned} C_H^2 \int_{-\infty}^{-1} \left( (t-s)^{H-1/2} - (-s)^{H-1/2} \right)^2 ds &= C_H^2 \int_1^{\infty} \left( (t+s)^{H-1/2} - s^{H-1/2} \right)^2 ds \\ &\leq C_H^2 t^2 (H-1/2)^2 \int_1^{\infty} s^{2H-3} ds = \frac{C_H^2 t^2 (H-1/2)^2}{2(1-H)} \quad \square \end{aligned}$$

## 6. A parallel algorithm for the approximation

In order to specify a concrete approximation of fBm, an input of the following type is given:

- A sequence of  $\ell$  time instances  $0 \leq t_1 < \dots < t_i < \dots < t_\ell \leq 1$ ;
- an approximation bound  $a > 0$ ; this is a probable upper bound on the distance between fBm and the approximation;
- a probability bound  $p > 0$  such that  $1 - p$  is a lower bound on the probability of correctness of the given approximation bound  $a$ .
- Optionally, the Hurst index  $H$  may be taken as an input, or may be viewed as fixed.

The algorithm has two parts. First, a preprocessing step uses  $a$  and  $p$  (as well as  $H$ ) to find the corresponding parameters  $N$  and  $q$ , in order to obtain the deviation



bound of Theorem 5.1. Second, a probabilistic parallel algorithm takes as inputs  $N$ ,  $q$ ,  $H$ , and sample times  $t_i$  ( $i = 1, \dots, \ell$ ); it outputs a sequence of numbers  $W(t_i, H, N)$  ( $i = 1, \dots, \ell$ ) such that  $\Pr\{\sup_{1 \leq i \leq \ell} |B_{t_i}^{(H)} - W(t_i, H, N)| \geq a\} \leq p$ .

### Part 1: Preprocessing step

On input  $a$ ,  $p$  (and  $H$ ) we find  $N$  and  $q$  by solving the system of two equations, obtained from Theorem 5.1:

$$(A) \quad a = \frac{C\sqrt{q}}{\sqrt{H(1-H)}} \frac{\sqrt{\log N}}{N^H}$$

$$(P) \quad p = \frac{3}{\sqrt{q}N^q}$$

This system is always solvable for  $p \in ]0, 1]$  and small enough  $a > 0$ , which can be seen as follows. We eliminate  $N$  and derive an equation with unknown  $q$ :

$$(Q) \quad a = \frac{C}{\sqrt{H(1-H)}} \cdot \frac{\sqrt{(1/q) \cdot \log(3/(p\sqrt{q}))}}{(3/(p\sqrt{q}))^{H/q}}$$

The numerator  $\sqrt{(1/q) \cdot \log(3/(p\sqrt{q}))}$  decreases strictly from  $\sqrt{\log(3/p)}$  (when  $q = 1$ ) to 0 (as  $q \rightarrow +\infty$ ). The denominator  $(3/(p\sqrt{q}))^{H/q}$  is bounded for  $q \in [1, +\infty]$  (it tends to 1 as  $q \rightarrow +\infty$ ), and is also bounded away from 0 for any given value of  $p$  and  $H$ . This implies that equation (Q) is solvable in  $q$  for any  $p \in ]0, 1]$ ,  $H \in ]0, 1[$  and  $a \in ]0, a_{\max}[$  (for some  $a_{\max} = a_{\max}(H, p) > 0$ ). Once a value for  $q$  has been found, equation (P) yields a value for  $N$  (which we round up to the nearest integer).

For the complexity analysis an upper bound on  $N$  in terms of  $a$  and  $p$  is needed.

**Proposition 6.1.** *Let  $N$  be the solution of equations (A) and (P). Then  $\log N \leq \frac{4}{H(2+H)} \cdot (|\log(pa)| + \log \frac{3c}{\sqrt{H(1-H)}})$ . Thus for fixed  $H$ ,  $\log N \leq O(|\log(pa)|)$ . Moreover,  $\log N \geq \frac{1}{H} \cdot (|\log a| + \log \frac{c}{\sqrt{H(1-H)}})$ , hence for fixed  $H$ ,  $\log N \geq \Omega(|\log a|)$ .*

**Proof.** We have  $N^\varepsilon > \sqrt{\log N}$ , for all  $\varepsilon > 0$ ,  $N > e$ . We will also need  $H > \varepsilon$ , so we pick  $\varepsilon = H/2$ . Equation (A) then implies  $N < \left( \frac{c\sqrt{q}}{\sqrt{H(1-H)}} \cdot \frac{1}{a} \right)^{2/H}$ .

Replacing  $\sqrt{q} = \frac{3}{pN^q}$  (from equation (P)), we obtain

$$N < \left( \frac{3c}{\sqrt{H(1-H)}} \cdot \frac{1}{pa} \right)^{2/H} \frac{1}{N^{qH/2}} < \left( \frac{3c}{\sqrt{H(1-H)}} \frac{1}{pa} \right)^{2/H} \frac{1}{N^{H/2}} \quad \text{since } 1 < q$$

Hence,

$$N^{1+H/2} < \left( \frac{3c}{\sqrt{H(1-H)}} \cdot \frac{1}{pa} \right)^{2/H}.$$

The first inequality then follows immediately. To obtain the last inequality we use  $1 < \sqrt{\log N}$  and  $1 < q$  in equation (A). This yields  $N^H > \frac{c}{\sqrt{H(1-H)}} \cdot \frac{1}{a}$ , from which the proposition follows immediately.  $\square$

**Part 2: Parallel algorithm to find approximants at sample times**

In the following parallel algorithm, only step 1 is probabilistic.

In this part, only  $N$  and  $\ell$  are used as complexity parameters. Since an approximation bound  $a$  is given, the numerical representations and calculations require an absolute precision  $\leq a$ . Hence binary or decimal expansions with  $\Theta(|\log a|)$  or more digits will be used for all numbers. We will see in the analysis of the algorithm below that a precision of  $\Theta(|\log a|)$  digits is sufficient. By the Proposition above,  $|\log a| \leq \Theta(\log N)$ .

**Algorithm**

Step 1 By using  $(2N + 3)\ell$  independent random processes in parallel, generate  $(2N + 3)\ell$  independent samples from the standard Gaussian distribution. Label the samples by  $\mathcal{L}(\omega_{3,i})$  and  $\mathcal{L}(\omega_{1,n,i})$ ,  $\mathcal{L}(\omega_{2,n,i})$ , for  $n = 0, \dots, N$ ,  $i = 1, \dots, \ell$ .

Step 2 By using  $(N + 1)\ell$  independent deterministic parallel processes, compute the wavelet coefficients  $\langle f_{t_i}^{(1)}, \mathcal{H}_n \rangle$  and  $\langle f_{t_i}^{(2)}, \mathcal{H}_n \rangle$  for  $i = 1, \dots, \ell$  and  $n = 0, \dots, N$ . Compute each  $\langle f_{t_i}^{(1)}, \mathcal{H}_n \rangle = 2^{j/2} \int_{2^{-j}k}^{2^{-j}(k+\frac{1}{2})} f^{(1)}(u)du - 2^{j/2} \int_{2^{-j}(k+\frac{1}{2})}^{2^{-j}(k+1)} f^{(1)}(u)du$ , and similarly, each  $\langle f_{t_i}^{(2)}, \mathcal{H}_n \rangle$ , by standard numerical integration techniques at a precision of  $\Theta(|\log a|)$ .

Step 3 By using deterministic parallel processes, compute for  $i = 1, \dots, \ell$ :

$$W_1(t_i, H, N)(\omega_{1,i}) = C_H \sum_{n=0}^N \langle f_{t_i}^{(1)}, \mathcal{H}_n \rangle \cdot \mathcal{L}(\omega_{1,n,i})$$

and

$$W_2(t_i, H, N)(\omega_{2,i}) = C_H \sum_{n=0}^N \langle f_{t_i}^{(2)}, \mathcal{H}_n \rangle \cdot \mathcal{L}(\omega_{2,n,i})$$

Here,  $\omega_{1,i} = (\omega_{1,n,i} : n = 0, \dots, N)$ , and  $\omega_{2,i} = (\omega_{2,n,i} : n = 0, \dots, N)$ .

Step 4 By using  $\ell$  deterministic parallel processes, compute for  $i = 1, \dots, \ell$ :

$$I_3(t_i, H)(\omega_{3,i}) = C_H \sqrt{\int_{-\infty}^{-1} ((t_i - s)^{H-1/2} - (-s)^{H-1/2})^2 ds} \cdot \mathcal{L}(\omega_{3,i})$$

Step 5 By using  $\ell$  deterministic parallel processes, compute for  $i = 1, \dots, \ell$ :

$$W(t_i, H, N)(\omega_i) = W_1(t_i, H, N)(\omega_{1,i}) + W_2(t_i, H, N)(\omega_{2,i}) + I_3(t_i, H)(\omega_{3,i})$$

where  $\omega_i = (\omega_{1,i}, \omega_{2,i}, \omega_{3,i})$ .

In step 1, each sample can be generated in constant time, so the parallel algorithm also takes constant time. In fact, samples from a standard Gaussian distribution can be pre-computed and stored for later use; in that case, step 1 does not need to be counted at all. In step 2, each wavelet coefficient can be computed in parallel time  $O(|\log a|)$ . All these coefficients are computed independently of each other, so the total parallel time is  $O(|\log a|) \leq O(\log N)$ . Step 3 carries out  $2\ell$  independent additions in parallel to find  $W_j(t_i, H, N)$ ,  $j = 1, 2$  and  $i = 1, \dots, \ell$ . For each  $W_j(t_i, H, N)$ ,  $N + 1$  terms of the form  $\langle f_{t_i}^{(j)}, \mathcal{H}_n \rangle \mathcal{L}(\omega_{j,n,1})$  are obtained first by independent multiplications, so they are obtained (with  $O(|\log a|)$  digits of precision) in parallel time  $O(|\log a|)$ . The  $N + 1$  terms are then added, which can be done by a parallel algorithm in time  $O(\log N) \cdot \alpha$ , where  $\alpha$  is the time used to add two items (see e.g., [14]). Addition of two  $b$ -bit numbers can be done in parallel time  $O(\log b)$  using  $O(b)$  processors (see e.g., [14]). Thus the parallel time complexity of step 3 is  $O(\log N \cdot \log \log N)$  using  $O(N \log N)$  processors. Steps 4 and 5 each use  $\ell$  independent parallel processes, and each process takes time  $O(|\log a|)$  for a precision of  $|\log a|$  digits. So the time complexity of these steps is  $O(\log N)$ . Therefore, the total time complexity of the above parallel algorithm is  $O(\log N \cdot \log \log N)$ , using  $O((\ell + \log N) N)$  processors.

By the above Proposition we conclude that for a fixed  $H$  the parallel time complexity is  $O(|\log(pa)| \cdot \log |\log(pa)|)$ .

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